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Galilean relativistic wave equations

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Abstract. The locally operating realisations of the Galilei group are studied. Lévy-Leblond's equation for spin-one-half Galilei relativistic massive systems is rederived by making use of group theoretical methods and a generalisation to spin one is proposed. Finally, the relation to the corresponding Poincaré relativistic equations is analysed.

1. Introduction

Several years ago Lévy-Leblond (1974) pointed out the convenience of a remodelling process in the logical (vs mathematical) structure of the foundations of the quantum theory. In particular, for several reasons, both epistemological and practical, he proposed that the 'correspondence principle', whose role in the historical development of the quantum theory has been very important, ought to be replaced by a 'symmetry principle' as a guide for building up the specific structure of such theory. We should like this paper to be one more step in this recasting process.

In a now classical paper, Wigner (1939) has shown that with the hypothesis of invariance under a group G of space–time transformations (kinematic group), any quantum system is described by means of the representation space of a (semi-unitary) projective representation of the group G. The free wave equation is but a method of specifying the corresponding representation, that is to say, the set of all solutions of the wave equation for a free system spans a representation space for the group G. If the (multiplier) representation is irreducible the quantum system is called elementary. We assume, with Lévy-Leblond (1967), that the system under consideration has no additional structure besides the one associated to the representation, taking this fact as a definition of 'elementarity'.

More recently, Hoogland (1967a, b, 1977) has shown that the classification of elementary systems according to the equivalence classes of (semi-unitary) projective representations of the kinematic group G cannot be considered as fully satisfactory. Only some realisations of G, called by him 'locally operating realisations' (LOR), must be considered as physically relevant. The important concept of equivalence of locally operating realisations is that of 'local equivalence', a refinement of the usual projective equivalence. From these ideas some light is shed on the appearance of the homogeneous group instead of the little group of Wigner's canonical theory for the characterisation of irreducible locally operating realisations.

In § 2 we explain briefly the concept of LOR and give some of the results of Hoogland, but they are presented in a slightly different form. We specialise to the case of the Galilei group and develop a method for comparing these realisations to the standard induced representations of the projective covering group $\bar{G}$ of G (Cariñena and Santander 1975), by means of a Fourier transformation.

Section 3 is devoted to the rederivation of the Galilean relativistic spin-$\frac{1}{2}$ massive wave equation proposed by Lévy-Leblond (1967). In this rederivation no 'linearising' process similar to the one used by Dirac is necessary. It may be worth remarking that we start with the locally operating realisation under which the equation we are looking for must be invariant, and then the explicit form of the inner product arises in a natural way as translated from the standard realisation. Two steps are to be made: a doubling of components in order to obtain a simpler expression for the inner product and a change of variables which permit us to decouple the components in two sets, each satisfying identical equations.

This method can be generalised in an easy way as is done in § 4 for the spin-one case. A comparison with the corresponding Poincaré relativistic equation is given in § 5.

We must remark that this rederivation is not to be confused with the more sophisticated one due to Niederer and O'Raifeartaigh (1977) where the equation is obtained from two representations of the homogeneous group and by making use of a projection operator. Nevertheless, the equations obtained by the method proposed here coincide with some of those derived by his methods.

2. Locally operating realisations of the Galilei group

We recall briefly the main concepts and ideas about LOR of kinematic symmetry groups.

Let G be a kinematic symmetry group (a connected Lie group) acting transitively on the space–time X, $x \xrightarrow{g} gx$. As is well known X can be identified with the homogeneous space G/K, K being the isotopy group of any fixed point $x_0 \in X$. Let $h: X \to \mathrm{G}$ be a Borel normalised section, i.e. a transformation h_x for each $x \in X$ such that $h_x x_0 = x$, with $h_{x_0} = e$. Once such a section h is given, every $g \in \mathrm{G}$ has a unique factorisation $g = h_{gx_0} \cdot \gamma(g)$, where $\gamma(g) \in \mathrm{K}$.

Following Hoogland (1976a, b) we use the term 'locally operating realisations' of G to mean the multiplier representations of G which operate locally in a space of multicomponent wavefunctions $\Psi: X \to V$, V being a (finite-dimensional) complex linear space, according to

$$[\mathscr{U}(g)\Psi](gx) = A(g, x)\Psi(x) \tag{2.1}$$

where $A(g, x)$ is a matrix called the gauge matrix of $\mathscr{U}$. The canonical topology in V induces a topology in the space of functions and we implicitly assume that the associated projective representation is continuous. This does not mean that $A: \mathrm{G} \times X \to \mathrm{GL}(n, \mathbb{C})$ itself is continuous, but at least A is a Borel function.

The natural concept of equivalence for LOR is not pseudo-equivalence of multiplier representations but the so-called local equivalence: two LOR of G, $\mathscr{U}$ and $\mathscr{U}'$ both operating on the same space of wavefunctions are called *locally equivalent* if there is a (Borel) function $\lambda: \mathrm{G} \to \mathrm{U}(1)$ and a linear transformation T in the space of wavefunctions such that

(i) $(T\Psi)(x) = S(x)\Psi(x)$

where $S(x)$ is a (Borel) non-singular matrix, i.e. T acts locally and

(ii) $\mathcal{U}'(g)=\lambda(g)T\mathcal{U}(g)T^{-1}$.

The local equivalence of LOR translates itself to some kind of equivalence between gauge matrices: two sets of gauge matrices are called *equivalent* if there exist (Borel) functions $\lambda:\mathrm{G}\to\mathrm{U}(1)$ and $S:X\to\mathrm{GL}(n,\mathbb{C})$ such that

$$A'(g,x)=\lambda(g)S(gx)A(g,x)S^{-1}(x).$$

Now we specialise G to be the Galilei group and X the corresponding Newtonian space–time. We adopt here the active viewpoint (see e.g. Fonda and Ghirardi 1970) according to which the element $g=(b,\boldsymbol{a},\boldsymbol{v},R)$ of $\mathcal{G}$ is the transformation of X given by

$$x\equiv\begin{pmatrix}\boldsymbol{x}\\ t\end{pmatrix}\to gx\equiv\begin{pmatrix}\boldsymbol{x}'\\ t'\end{pmatrix}=\begin{pmatrix}R\boldsymbol{x}+\boldsymbol{v}t+\boldsymbol{a}\\ t+b\end{pmatrix} \tag{2.2}$$

$(\boldsymbol{x},t)$ and $(\boldsymbol{x}',t')$ being respectively the coordinates in some fixed inertial frame of an event and its image under g. We remark that this action is transitive.

The specific semi-direct structure of the homogeneous Galilei group greatly simplifies the results about local equivalence classes of LOR of $\mathcal{G}$. As usual, we take

$$x_0=(\mathbf{0},0)\qquad h_x=(t,\boldsymbol{x},0,\mathbb{1})\qquad \gamma(g)=(0,\mathbf{0},\boldsymbol{v},R)\equiv(\boldsymbol{v},R) \tag{2.3}$$

and then it can be shown that any gauge matrix is equivalent to a gauge matrix of the form

$$A(g,x)=\omega(g,h_x)\mathcal{D}(\gamma(g))$$

where ω is a factor system of $\mathcal{G}$ and $\mathcal{D}$ is a multiplier matrix representation of K with factor system $\omega|_{\mathrm{K}\times\mathrm{K}}$. Conversely, if ω and $\mathcal{D}$ are given as above, then $A(g,x)$ defined by the former expression is a gauge matrix. The interplay between ω and $\mathcal{D}$ in order to give equivalent gauge matrices lies at the heart of the concept of super-equivalence of group exponents, but we do not enter into this question here.

As is known any factor system of $\mathcal{G}$ is equivalent to some standard $\omega_{\lambda,l}$, $\lambda\in\mathbb{R}$, $l\in\{1,-1\}$, given by (Brennich 1970, Cariñena and Santander 1975)

$$\omega_{\lambda,l}(g',g)=\exp[\mathrm{i}\lambda(\tfrac{1}{2}\boldsymbol{v}'^2b+\boldsymbol{v}'\cdot R'\boldsymbol{a})]\zeta_l(R',R). \tag{2.4}$$

The most general (up to local equivalence) multiplier LOR of $\mathcal{G}$ is

$$[\mathcal{U}(g)\Psi](\boldsymbol{x}'t')=\exp[\mathrm{i}\lambda(\tfrac{1}{2}\boldsymbol{v}^2t+\boldsymbol{v}\cdot R\boldsymbol{x})]\mathcal{D}(\boldsymbol{v},R)\Psi(\boldsymbol{x},t) \tag{2.5}$$

whose ingredients are (i) the factor system of $\mathcal{G}$ given by (2.4), (ii) the matrix multiplier representation $\mathcal{D}$ of K with factor system $\xi(\boldsymbol{v}',R';\boldsymbol{v},R)=\zeta_l(R',R)$.

For practical purposes we notice that because of the structure of the factor system of $\mathcal{D}$, this representation can always be obtained from a linear representation of the universal covering group K* of K but this fact follows from the formalism and has not to be assumed in advance.

The preceding considerations explain why we can restrict our attention to those LOR of G given by (2.5). However, we are interested in the unitary irreducible representations and then in order to achieve this, the following three steps are to be made.

(i) We must pick out some subspace $\mathcal{F}$ of functions $\Psi:X\to V$ in such a way that it is irreducible under the action (2.5).

(ii) Simultaneously we must choose the representation $\mathcal{D}$ of K.

(iii) We must endow the representation space with an inner product structure in such a way that the LOR (2.5) is unitary.

The two first steps are interrelated because V is the representation space of $\mathscr{D}$.

We shall make these steps with the help of the Fourier transformation which enables us to compare the LOR under consideration to the standard theory of induced representations (see e.g. Mackey 1958, Simms 1968) of $\mathscr{G}$.

The space–time has been identified to T_4 (space–time translations group) by means of the choice of the point x_0 and the section h. Let $\Phi(k)$ be the Fourier transform of $\Psi(x)$,

$$\Phi(\boldsymbol{p}, E)=\frac{1}{(2\pi)^2}\int \exp[\mathrm{i}(Et-\boldsymbol{px})]\Psi(\boldsymbol{x}, t)\,\mathrm{d}^3\boldsymbol{x}\,\mathrm{d}t \tag{2.6}$$

where $k=(\boldsymbol{p}, E)$. Then, the transformation law of $\Phi(k)$ under the action of $\mathscr{G}$ is

$$[\mathscr{U}(g)\Phi](\boldsymbol{p}, E)=\exp[\mathrm{i}(Eb-\boldsymbol{pa})]\mathscr{D}(\boldsymbol{v}, R)\Phi(\boldsymbol{p}', E') \tag{2.7}$$

where

$$\boldsymbol{p}'=R^{-1}(\boldsymbol{p}-\lambda\boldsymbol{v}) \qquad E'=E+\tfrac{1}{2}\lambda\boldsymbol{v}^2-\boldsymbol{pv}. \tag{2.8}$$

We remark that for $\lambda\neq 0$, $2\lambda E-\boldsymbol{p}^2=2\lambda E'-\boldsymbol{p}'^2$, and this relation shows a necessary condition for irreducibility of the corresponding LOR, the vanishing of the function $\Phi(p, E)$ when $2\lambda E-\boldsymbol{p}^2$ is not equal to an arbitrary but fixed real number, say ρ. Furthermore, the only physically interesting case is $\lambda\neq 0$ as we are going to see shortly.

Now we should like to compare the preceding realisation with the canonical realisations of $\mathscr{G}$, whose theory is developed e.g. in Lévy-Leblond (1963, 1972) or Cariñena and Santander (1975). We recall that all irreducible multiplier representations (up to projective equivalence) of $\mathscr{G}$ can be obtained from the irreducible representations (up to pseudo-equivalence) of an eleven-parameter group, called the projective covering group $\bar{\mathscr{G}}$ of $\mathscr{G}$ (or extended Galilei group). In particular, the physically relevant irreducible representations of $\bar{\mathscr{G}}$ are the ones labelled by $[m, U, s]$, explicitly given by

$$\begin{aligned}&[[m, U, s](\theta, b, \boldsymbol{a}, \boldsymbol{v}, A)\psi](\boldsymbol{p})\\&\quad=\mathrm{e}^{\mathrm{i}m\theta}\exp\{\mathrm{i}[(\boldsymbol{p}^2/2m+U)b-\boldsymbol{pa}]\}D_s(A)\psi(R^{-1}(\boldsymbol{p}-m\boldsymbol{v})).\end{aligned} \tag{2.9}$$

The parameters $m\neq 0$ and s are to be interpreted as the mass and the spin of the elementary system. The case $m=0$ corresponding to 'almost' linear representations is unphysical (Inonu and Wigner 1952, Hamermesh 1960).

Now we must compare the realisation (2.7) obtained from the original LOR (2.5) with that of (2.9): the similarity of the factor systems shows that λ has to be identified to the mass of the elementary system and ρ corresponds to the internal energy. No restriction is imposed by taking $\rho=0$ because the projective equivalence involved acts locally on $\Psi(\boldsymbol{x}, t)$. Notice, however, that sometimes other choices are more natural: for instance $\rho=m$ when studying 'non-relativistic' limits. In this way we also obtain a first necessary condition for the irreducibility of the original LOR

$$(2mE-\boldsymbol{p}^2)\Phi(\boldsymbol{p}, E)=0 \tag{2.10}$$

which when translated to the LOR gives a (multi-component) Schrödinger equation

$$\mathrm{i}\,\partial_t\Psi(\boldsymbol{x}, t)=-(\boldsymbol{\nabla}^2/2m)\Psi(\boldsymbol{x}, t).$$

Let us now define for any function $\Phi(\boldsymbol{p}, E)$ a new function $\phi(\boldsymbol{p}, E)$ by means of

$$\phi(\boldsymbol{p}, E) = \mathscr{D}(-\boldsymbol{p}/m, \mathbb{1})\Phi(\boldsymbol{p}, E). \tag{2.11}$$

The associated realisation of $\mathscr{G}$ (see (2.8)) is given by

$$[U(g)\phi](\boldsymbol{p}, E) = \exp[\mathrm{i}(Eb - \boldsymbol{pa})]\mathscr{D}(\boldsymbol{0}, R)\phi(\boldsymbol{p}', E'). \tag{2.12}$$

From (2.5) and (2.12) we realise that when we want to describe a spin-s elementary system by means of a LOR (2.5), the restriction of the multiplier representation $\mathscr{D}$ of K to SO(3) must contain a single D_s representation (although possibly repeated) and no other representations. This condition narrows the range of possible candidates for the representation $\mathscr{D}$.

3. Galilean wave equations for spin $\frac{1}{2}$

When we wish to describe a spin-one-half massive elementary system (under Galilean relativity) by means of a LOR (2.5) of $\mathscr{G}$, we must start with a multiplier representation of K whose restriction to SO(3) is (a multiple of) $D_{1/2}$. The most obvious choice is $D_{1/2}$ itself, which is a representation as a consequence of the semidirect structure of K. It is more interesting, however, to take a faithful representation. The lowest dimension of a faithful representation is four, and then the requirement $\mathscr{D}|_{\mathrm{SO}(3)} \approx D_{1/2} \oplus D_{1/2}$ determines $\mathscr{D}$ (up to equivalence) to be

$$\mathscr{D}(\boldsymbol{v}, R) \equiv \Delta^{1/2}(\boldsymbol{v}, R) = \begin{pmatrix} D_{1/2}(R) & 0 \\ \frac{1}{2}\boldsymbol{\sigma} \cdot \boldsymbol{v} D_{1/2}(R) & D_{1/2}(R) \end{pmatrix}. \tag{3.1}$$

Following the steps which have been previously explained one obtains an irreducible LOR of $\mathscr{G}$ by means of four-component functions Ψ satisfying $\mathrm{i}\partial_t\Psi(\boldsymbol{x}, t) = -(\boldsymbol{\nabla}^2/2m)\Psi(\boldsymbol{x}, t)$ and transforming under $\mathscr{G}$ as follows:

$$[\mathscr{U}(g)\Psi](\boldsymbol{x}', t') = \exp[\mathrm{i}m(\tfrac{1}{2}\boldsymbol{v}^2 t + \boldsymbol{v} \cdot R\boldsymbol{x})]\Delta^{1/2}(\boldsymbol{v}, R)\Psi(\boldsymbol{x}, t). \tag{3.2}$$

Now the following question arises: is there an inner product in this representation space such that this realisation is unitary? A direct answer is not immediate because of the not fully reducible nature of $\Delta^{1/2}$. But a glance to the transformation law (2.12) of the associated realisation of $\mathscr{G}$ by ϕ functions makes it obvious that the ϕ functions transform unitarily if the inner product is defined as

$$\langle\phi_1|\phi_2\rangle = \int \phi_1^+(\boldsymbol{p})\phi_2(\boldsymbol{p})\,\mathrm{d}^3\boldsymbol{p} \tag{3.3}$$

where the shorthand notation $\phi(\boldsymbol{p}) = \phi(\boldsymbol{p}, E = \boldsymbol{p}^2/2m)$ has been used. (The measure $\mathrm{d}^3\boldsymbol{p}$ is the invariant measure on the paraboloid $2mE - \boldsymbol{p}^2 = 0$.) This inner product can be translated to the realisations (2.7) and (2.5). The translation to (2.7) is simple and gives (with the same shorthand)

$$\langle\Phi_1|\Phi_2\rangle = \int \Phi_1^+(\boldsymbol{p})G(\boldsymbol{p})\Phi_2(\boldsymbol{p})\,\mathrm{d}^3\boldsymbol{p} \tag{3.4}$$

where $G(\boldsymbol{p})$ is the matrix

$$G(\boldsymbol{p}) = \mathscr{D}^+(-\boldsymbol{p}/m, \mathbb{1})\mathscr{D}(-\boldsymbol{p}/m, \mathbb{1}) = \begin{pmatrix} 1 + (\boldsymbol{\sigma} \cdot \boldsymbol{p})^2/(2m)^2 & -\boldsymbol{\sigma} \cdot \boldsymbol{p}/2m \\ -\boldsymbol{\sigma} \cdot \boldsymbol{p}/2m & 1 \end{pmatrix}. \tag{3.5}$$

The p dependence of this metric matrix makes the expression of the inner product by means of the wavefunctions $\Psi(x, t)$ very complicated. It is convenient to make some manipulations before translating the inner product, in order to produce simpler expressions.

For each function Φ one can define a new function Θ by $\Theta(p) = G(p)\Phi(p)$. The eight-component functions $\hat{\Phi} = \binom{\Phi}{\Theta}$ support a new realisation of $\mathscr{G}$. If we write $\Phi = (\varphi_1, \varphi_2)$, $\Theta = (\theta_1, \theta_2)$, the relations between Θ and Φ are given by

$$\left.\begin{aligned}[1+(\boldsymbol{\sigma}\cdot\boldsymbol{p})^2/(2m)^2]\varphi_1-(\boldsymbol{\sigma}\cdot\boldsymbol{p}/2m)\varphi_2=\theta_1\\ -(\boldsymbol{\sigma}\cdot\boldsymbol{p}/2m)\varphi_1+\varphi_2=\theta_2\end{aligned}\right\}$$

$$\left.\begin{aligned}\theta_1+(\boldsymbol{\sigma}\cdot\boldsymbol{p}/2m)\theta_2=\varphi_1\\ (\boldsymbol{\sigma}\cdot\boldsymbol{p}/2m)\theta_1+[1+(\boldsymbol{\sigma}\cdot\boldsymbol{p})^2/(2m)^2]\theta_2=\varphi_2\end{aligned}\right\} \tag{3.6}$$

and the inner product (3.4) has the expression

$$\langle\hat{\Phi}_1|\hat{\Phi}_2\rangle = \int (\varphi_1^+(\boldsymbol{p})\theta_1(\boldsymbol{p}) + \varphi_2^+(\boldsymbol{p})\theta_2(\boldsymbol{p}))\,\mathrm{d}^3\boldsymbol{p}.$$

We can use the relations (3.6) to recast this inner product in the form

$$\int \{\varphi_1^+(\boldsymbol{p})[\varphi_1(\boldsymbol{p})-(\boldsymbol{\sigma}\cdot\boldsymbol{p}/2m)\theta_2(\boldsymbol{p})]+[\theta_2^+(\boldsymbol{p})+(\boldsymbol{\sigma}\cdot\boldsymbol{p}/2m)\varphi_1^+(\boldsymbol{p})]\theta_2(\boldsymbol{p})\}\,\mathrm{d}^3\boldsymbol{p}$$

$$= \int \{\varphi_1^+(\boldsymbol{p})\varphi_1(\boldsymbol{p}) + \theta_2^+(\boldsymbol{p})\theta_2(\boldsymbol{p})\}\,\mathrm{d}^3\boldsymbol{p}. \tag{3.7}$$

Then, in order to obtain this simple expression for the inner product we have doubled the number of components. We can now consider the following change of variables:

$$\left.\begin{aligned}\varphi=\varphi_1\\ \chi=\varphi_2-\theta_2\end{aligned}\right\} \qquad \left.\begin{aligned}\varphi'=\theta_2\\ \chi'=\varphi_1-\theta_1\end{aligned}\right\} \tag{3.8}$$

The new functions will be related by

$$\left.\begin{aligned}[(\boldsymbol{\sigma}\cdot\boldsymbol{p})^2/2m]\varphi-\boldsymbol{\sigma}\cdot\boldsymbol{p}\chi=0\\ -\boldsymbol{\sigma}\cdot\boldsymbol{p}\varphi+2m\chi=0\end{aligned}\right\} \qquad \left.\begin{aligned}[(\boldsymbol{\sigma}\cdot\boldsymbol{p})^2/2m]\varphi'-\boldsymbol{\sigma}\cdot\boldsymbol{p}\chi'=0\\ -\boldsymbol{\sigma}\cdot\boldsymbol{p}\varphi'+2m\chi'=0\end{aligned}\right\} \tag{3.9}$$

and the inner product may be rewritten as

$$\langle\hat{\Phi}_1|\hat{\Phi}_2\rangle = \int \varphi^+(\boldsymbol{p})\varphi(\boldsymbol{p})\,\mathrm{d}^3\boldsymbol{p} + \int \varphi'^+(\boldsymbol{p})\varphi'(\boldsymbol{p})\,\mathrm{d}^3\boldsymbol{p}. \tag{3.10}$$

If $\Xi(\boldsymbol{p}) = \binom{\varphi(\boldsymbol{p})}{\chi(\boldsymbol{p})}$, a simple algebraic calculation gives the following transformation law for Ξ

$$[\mathscr{U}(g)\Xi](\boldsymbol{p}) = \Delta^{1/2}(\boldsymbol{v}, R)\Xi(R^{-1}(\boldsymbol{p}-m\boldsymbol{v})) \tag{3.11}$$

and an identical transformation law for Ξ'. In the calculation we must make use of the relation $D_{1/2}(R)\boldsymbol{\sigma}\cdot\boldsymbol{w}(D_{1/2}(R))^{-1} = \boldsymbol{\sigma}\cdot R\boldsymbol{w}$.

Then, relations (3.9) and (3.11) clearly show that the linear change (3.8) produces a decoupling of components in such a way that the realisation by $\hat{\Phi}$ appears as a direct sum of twice the same realisation by Ξ and Ξ'. The irreducibility conditions $(2mE-\boldsymbol{p}^2)\Phi(\boldsymbol{p}, E) = 0$ imply of course identical conditions for Ξ and Ξ'.

So, we have an irreducible realisation of $\mathscr{G}$ by means of the functions $\Xi(\boldsymbol{p}) = \binom{\varphi(\boldsymbol{p})}{\chi(\boldsymbol{p})}$ which satisfy the support condition and the equations (3.9), and which is unitary with respect to the inner product given by

$$\langle \Xi_1 | \Xi_2 \rangle = \int \varphi_1^+(\boldsymbol{p}) \varphi_2(\boldsymbol{p}) \, \mathrm{d}^3 \boldsymbol{p}. \tag{3.12}$$

Now the presence of $(\boldsymbol{\sigma} \cdot \boldsymbol{p})^2 = \boldsymbol{p}^2$ in the equation (3.9*a*) makes it possible to incorporate the irreducibility condition by writing it in the form

$$\begin{aligned} &E\varphi(\boldsymbol{p}, E) - \boldsymbol{\sigma} \cdot \boldsymbol{p}\chi(\boldsymbol{p}, E) = 0 \\ &-\boldsymbol{\sigma} \cdot \boldsymbol{p}\varphi(\boldsymbol{p}, E) + 2m\chi(\boldsymbol{p}, E) = 0 \end{aligned} \tag{3.13}$$

which immediately implies the support conditions for both φ and χ.

Finally, we return to the description by wavefunctions on space–time by means of the inverse Fourier transformation. The irreducible, unitary LOR is given in terms of the wavefunctions $\Psi(\boldsymbol{x}, t) = \binom{\varphi(\boldsymbol{x}, t)}{\chi(\boldsymbol{x}, t)}$ satisfying the equations

$$\begin{aligned} &\mathrm{i}\partial_t \varphi(\boldsymbol{x}, t) + \mathrm{i}(\boldsymbol{\sigma} \cdot \boldsymbol{\nabla})\chi(\boldsymbol{x}, t) = 0 \\ &\mathrm{i}(\boldsymbol{\sigma} \cdot \boldsymbol{\nabla})\varphi(\boldsymbol{x}, t) + 2m\chi(\boldsymbol{x}, t) = 0 \end{aligned} \tag{3.14}$$

and transforming under $\mathscr{G}$ as follows:

$$[\mathscr{U}(g)\Psi](\boldsymbol{x}', t') = \exp[\mathrm{i}m(\tfrac{1}{2}\boldsymbol{v}^2 t + \boldsymbol{v} \cdot R\boldsymbol{x})]\Delta^{1/2}(\boldsymbol{v}, R)\Psi(\boldsymbol{x}, t).$$

The inner product making unitary this LOR is

$$\langle \Psi_1 | \Psi_2 \rangle = \int \varphi_1^+(\boldsymbol{x}, t)\varphi_2(\boldsymbol{x}, t) \, \mathrm{d}^3\boldsymbol{x}$$

which is positive definite for solutions of the wave equation (3.14).

The wave equations (3.13) and (3.14) are just the equations first obtained by Lévy-Leblond (1967) by means of the heuristic idea used by Dirac in the derivation of his equation. Notice that the equations (3.14) differ from the original of Lévy-Leblond through the sign in front of $\boldsymbol{\sigma} \cdot \boldsymbol{\nabla}$ which can be absorbed by a redefinition of χ and by a change of $\Delta^{1/2}$ to another equivalent representation, namely, that obtained from $\Delta^{1/2}$ by a change of sign in front of $\boldsymbol{\sigma} \cdot \boldsymbol{v}$.

We remark particularly the natural appearance of the inner product as translated from the ϕ realisation, as well as the procedure of obtaining the equation starting from the representation $\Delta^{1/2}$ of the homogeneous group which makes it Galilean invariant.

4. Galilean wave equation for spin one

The preceding method can be easily generalised in order to obtain a wave equation describing spin-one massive elementary systems.

We start from the (linear) representation of K given by

$$\Delta^1(\boldsymbol{v}, R) = \begin{pmatrix} D_1(R) & 0 \\ \boldsymbol{S} \cdot \boldsymbol{v} D_1(R) & D_1(R) \end{pmatrix} \tag{4.1}$$

where $\boldsymbol{S}$ are the generators of the representation D_1 of SO(3). To obtain the corresponding LOR of $\mathscr{G}$ one must consider the linear space of six-component functions

$\Psi(\boldsymbol{x}, t)$, transforming under $\mathscr{G}$ by

$$[\mathscr{U}(g)\Psi](\boldsymbol{x}', t') = \exp[\mathrm{i}m(\tfrac{1}{2}\boldsymbol{v}^2 t + \boldsymbol{v} \cdot R\boldsymbol{x})]\Delta^1(\boldsymbol{v}, R)\Psi(\boldsymbol{x}, t) \tag{4.2}$$

and satisfying the multicomponent Schrödinger wave equation

$$\mathrm{i}\, \partial_t \Psi(\boldsymbol{x}, t) = -(\nabla^2/2m)\Psi(\boldsymbol{x}, t). \tag{4.3}$$

This realisation is unitary if a convenient inner product is defined. So, we introduce $\phi(\boldsymbol{p}, E) = \Delta^1(-\boldsymbol{p}/m, \mathbb{1})\Phi(\boldsymbol{p}, E)$ where $\Phi(\boldsymbol{p}, E)$ is, as before, the Fourier transform of $\Psi(\boldsymbol{x}, t)$. These functions transform under $\mathscr{G}$ by

$$[\mathscr{U}(g)\phi](\boldsymbol{p}, E) = \exp[\mathrm{i}(Eb - \boldsymbol{pa})]\Delta^1(\boldsymbol{0}, R)\phi(\boldsymbol{p}', E') \tag{4.4}$$

(see (2.8) and (2.12)), and the definition (with the usual shorthand)

$$\langle \phi_1 | \phi_2 \rangle = \int \phi_1^+(\boldsymbol{p})\phi_2(\boldsymbol{p})\, \mathrm{d}^3\boldsymbol{p} \tag{4.5}$$

makes unitary the representation (4.4). When this inner product is transported to the realisation of $\mathscr{G}$ by $\Phi(\boldsymbol{p}, E)$ functions, a matrix $G(\boldsymbol{p})$ arises in the inner product

$$G(\boldsymbol{p}) = \begin{pmatrix} 1 + \boldsymbol{S} \cdot \boldsymbol{p}/m^2 & -\boldsymbol{S} \cdot \boldsymbol{p}/m \\ -\boldsymbol{S} \cdot \boldsymbol{p}/m & 1 \end{pmatrix}. \tag{4.6}$$

Now, if we define $\Theta(\boldsymbol{p}) = G(\boldsymbol{p})\Phi(\boldsymbol{p})$, after the linear change with the same expression as (3.8) one obtains a pair of decoupled wave equations in momentum space

$$\left.\begin{aligned} (\boldsymbol{S} \cdot \boldsymbol{p})^2 \varphi - m(\boldsymbol{S} \cdot \boldsymbol{p})\chi = 0 \\ -m(\boldsymbol{S} \cdot \boldsymbol{p})\varphi + m^2\chi = 0 \end{aligned}\right\} \qquad \left.\begin{aligned} (\boldsymbol{S} \cdot \boldsymbol{p})^2 \varphi' - m(\boldsymbol{S} \cdot \boldsymbol{p})\chi' = 0 \\ -m(\boldsymbol{S} \cdot \boldsymbol{p})\varphi' + m^2\chi' = 0 \end{aligned}\right\}. \tag{4.7}$$

We remark that the first of each set of equations is a consequence of the second (as it already was in the spin-$\frac{1}{2}$ case). But there is a difference with respect to the spin-$\frac{1}{2}$ case: the support condition for φ and χ cannot be immediately incorporated in (4.7) in a reasonable way that makes it unnecessary. Anyway, if we multiply the first equation by $\boldsymbol{S} \cdot \boldsymbol{p}$ and use $(\boldsymbol{S} \cdot \boldsymbol{p})^3 = \boldsymbol{p}^2(\boldsymbol{S} \cdot \boldsymbol{p})$ we obtain

$$\begin{aligned} &2E(\boldsymbol{S} \cdot \boldsymbol{p})\varphi(\boldsymbol{p}, E) - (\boldsymbol{S} \cdot \boldsymbol{p})^2\chi(\boldsymbol{p}, E) = 0 \\ &-(\boldsymbol{S} \cdot \boldsymbol{p})\varphi(\boldsymbol{p}, E) + m\chi(\boldsymbol{p}, E) = 0 \end{aligned} \tag{4.8}$$

equations which automatically imply the support condition for χ but only the weaker condition for φ, $(2mE - \boldsymbol{p}^2)(\boldsymbol{S} \cdot \boldsymbol{p})\varphi = 0$. Thus, the condition $(2mE - \boldsymbol{p}^2)\varphi = 0$ must be added to the system (4.8). Nevertheless, the form (4.8) allows for an easy comparison to the corresponding spin-one wave equation as we shall see in the next section.

The transformation law of the wavefunctions $\Xi = \binom{\varphi}{\chi}$ is easily found to be

$$[\mathscr{U}(g)\Xi](\boldsymbol{p}) = \Delta^1(\boldsymbol{v}, R)\Xi(R^{-1}(\boldsymbol{p} - m\boldsymbol{v})) \tag{4.9}$$

which is unitary with respect to the inner product

$$\langle \Xi_1 | \Xi_2 \rangle = \int \varphi_1^+(\boldsymbol{p})\varphi_2(\boldsymbol{p})\, \mathrm{d}^3\boldsymbol{p}.$$

The corresponding LOR by wavefunctions on the space–time is obtained by means of the inverse Fourier transformation.

We remark that the procedure suggested by Niederer and O'Raifeartaigh (1977), when applied to the representations Δ^1, $(\Delta^1)^{\dagger-1}$ and the projection operator $W = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ also leads to the wave equation (4.7), although the method is different.

5. Connection with the $\mathcal{P}$ relativistic wave equations

The relation between the $\mathcal{G}$ relativistic wave equation for spin-$\frac{1}{2}$ and the well known Dirac equation has been shown by Lévy-Leblond (1967). Our aim is to show the corresponding relation in the spin-one case. Many such spin-one wave equations can be found in the physical literature, and therefore it is natural to choose the $\mathcal{P}$ relativistic equation obtained in a similar way in order to compare both equations.

For the Poincaré group $\mathcal{P} = \{(a, \Lambda)\}$, we take, as usual,

$$x_0 = (\mathbf{0}, 0) \qquad h_x = ((t, \boldsymbol{x}), \mathbb{1}) \qquad \gamma(a, \Lambda) = (0, \Lambda)$$

and furthermore we know that any factor system of $\mathcal{P}$ is equivalent to some standard ω_l, $l \in \{1, -1\}$, given by

$$\omega_l(g', g) = \zeta_l(R', R).$$

The results stated in § 2 immediately give the most general (up to equivalence) LOR of $\mathcal{P}$, as

$$[\mathcal{U}(a, \Lambda)\Psi](\Lambda x + a) = \mathcal{D}(\Lambda)\Psi(x) \tag{5.1}$$

where $\mathcal{D}$ is a multiplier matricial representation of the Lorentz group with factor system ζ_l. That representation can be obtained from a linear representation D of SL(2, $\mathbb{C}$), so that (5.1) is equivalent to the more familiar form

$$[\mathcal{U}(a, \Lambda)\Psi](\Lambda x + a) = D(A_\Lambda)\Psi(x). \tag{5.2}$$

If $\Phi(p)$ is the Fourier transform of $\Psi(x)$, the support condition $(p^2 - m^2)\Phi(p) = 0$ appears as a necessary condition in order to obtain an irreducible LOR corresponding to massive elementary $\mathcal{P}$ relativistic systems. The restriction to positive energy, given by $\theta(p^0)\Phi(p) = 0$ leads to a dispersion relation (Boya 1970) in $\Psi(x)$, but this restriction will not be considered below.

If $\hat{p} = (\mathbf{0}, m)$ is a selected point on the orbit Ω_m, take $L(p)$ (on Ω_m) as the pure Lorentz transformation mapping $\hat{p}$ on the point p

$$A_{L(p)} = [m(p^0 + m)]^{-1/2}[(p^0 + m) + \boldsymbol{\sigma} \cdot \boldsymbol{p}]$$

and define $\phi(p) = \mathcal{D}[L^{-1}(p)]\Phi(p)$; then the ϕ functions transform under $\mathcal{P}$ as follows:

$$[\mathcal{U}(a, \Lambda)\phi](p) = e^{-ipa}Q(p, \Lambda)\phi(\Lambda^{-1}p)$$

where $Q(p, \Lambda) = \mathcal{D}[L^{-1}(p)\Lambda L(\Lambda^{-1}p)]$. If D is an irreducible representation $D_{j,j'}$ of SL(2, $\mathbb{C}$) (with both j and j' non-zero), the restriction to SU(2) is a direct sum of different representations D_j of SU(2), so that additional restrictive conditions are necessary to eliminate the superfluous wrong spins. This does not occur if the representations $D_{s,0}$ or $D_{0,s}$ are chosen (Pursey 1965).

The connection with the Wigner canonical realisation enables us to introduce an inner product in such a way that the realisation is unitary. This inner product is in our case

$$\langle\Phi_1|\Phi_2\rangle = \int \Phi_1^+(p)\mathscr{D}[L^{-2}(p)]\Phi_2(p)\frac{d^3\boldsymbol{p}}{(\boldsymbol{p}^2+m^2)^{1/2}}. \tag{5.3}$$

A simpler inner product is obtained following the same steps than in the Galilean case. We make a doubling of components according to

$$\varphi(p)=\Phi(p) \qquad \chi(p)=\mathscr{D}[L^{-2}(p)]\Phi(p).$$

In particular, if $s=\frac{1}{2}$, the connection equations read

$$(p^0-\boldsymbol{\sigma}\cdot\boldsymbol{p})\varphi = m\chi \qquad (p^0+\boldsymbol{\sigma}\cdot\boldsymbol{p})\chi = m\varphi$$

with χ, φ being covariant objects transforming under $D_{\frac{1}{2},0}$ and $D_{0,\frac{1}{2}}$ respectively. In the non-relativistic limit ($p^0=m+E_{\rm nr}$, $|\boldsymbol{p}|/m \ll 1$) the upper and lower components coincide. The linear changes $\varphi'=2^{-1/2}(\varphi+\chi)$, $\chi'=2^{-1/2}(\varphi-\chi)$ leads to the usual form of the Dirac equation, whose non-relativistic limit is just the equation proposed by Lévy-Leblond.

The case $s=1$ gives

$$\begin{aligned}&[m^2+2p^0(\boldsymbol{S}\cdot\boldsymbol{p})+2(\boldsymbol{S}\cdot\boldsymbol{p})^2]\chi = m^2\varphi\\ &[m^2-2p^0(\boldsymbol{S}\cdot\boldsymbol{p})+2(\boldsymbol{S}\cdot\boldsymbol{p})^2]\varphi = m^2\chi\end{aligned} \tag{5.4}$$

for the connection equations with φ and χ now transforming under $D_{1,0}$ and $D_{0,1}$. These equations are the ones proposed by Sankaranayanan and Good (1965), but the derivation of these equations by them uses other methods. The former wave equations can be written in the form

$$\{D_{1,0}\oplus D_{0,1}\}L^{-2}(p)\Psi = \beta\Psi$$

where $\beta=\begin{pmatrix}0&1\\1&0\end{pmatrix}$. In terms of $\varphi'=2^{-1/2}(\varphi+\chi)$ and $\chi'=2^{-1/2}(\varphi-\chi)$ the spin-one equations read

$$\begin{aligned}&(\boldsymbol{S}\cdot\boldsymbol{p})^2\varphi'-p^0(\boldsymbol{S}\cdot\boldsymbol{p})\chi'=0\\ &p^0(\boldsymbol{S}\cdot\boldsymbol{p})\varphi'-[m^2+(\boldsymbol{S}\cdot\boldsymbol{p})^2]\chi'=0\end{aligned} \tag{5.5}$$

whose non-relativistic limit gives in a simple way the spin-one 'non-relativistic' equations proposed in § 4.

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